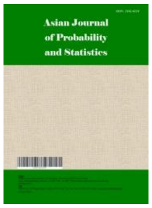

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On The Transmuted Powered Moment Exponential Distribution

Zafar Iqbal[1*], Muhammad Rashad[2], Abdur Razaq[3], Muhammad Salman[3] and Afsheen Javed[4]

[1]Department of Statistics, Government College, Gujranwala, Pakistan.
[2]Superior University, Lahore, Pakistan.
[3]Crop Reporting Service, Department of Agriculture, Pakistan.
[4]Punjab College, Gujrat, Pakistan.

Authors' contributions

This work was carried out in collaboration among all authors. Authors ZI and MR designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AR and MS managed the analyses of the study. Author AJ managed the literature searches. All authors read and approved the final manuscript.

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Abstract

We introduce a new class of lifetime models called the transmuted powered moment exponential distribution**.** More specifically, the transmuted powered moment exponential distribution covers several new distributions. Survival analysis including survival function, hazard rate function and other related measures are computed. Analytical expressions for various mathematical properties of TPMED including r^{th} moment, quantile function, inequality measures, and parameters are estimated by using maximum likelihood estimation and order statistics are also derived. A simulation study of the proposed distribution is performed. It is discovered that the Maximum Likelihood Estimators are consistent since the bias and Mean Square Error approach to zero when the sample size increases. The usefulness of the model associated with this distribution is illustrated by two real data sets and the new model provides a better fit than the models provided in literature.

**Corresponding author: Email: rashid.geostat@gmail.com;*

Keywords: Transmuted power moment exponential distribution (TPMED); r^{th} moment quantile function; inequality measures; maximum likelihood estimation and order statistics.

1 Introduction

An interesting method of adding a new parameter to an existing distribution that would offer more distributional flexibility [1-8]. We introduce a new class of lifetime models called the transmuted powered moment exponential distribution. The statistical development in the field of distribution theory [9-11] has long history but the forward-thinking in this field is transmuted techniques done by Shaw and Buckley (2009). Some mathematical properties of TPME distribution [12-14] including rth moment, quantile function, inequality measures, and parameters are estimated by using maximum likelihood estimation and order statistics are also derived [15,16]. Simulation study of proposed distribution is performed. Many statisticians gave some support of probability distributions in favour of above technique like Patel [17], Mir [18], Mankhdum and Nasiri [19], .Dara and Ahmed [20], Kharazmi et al., [21], Dey et al., [22], Okorie et al., [23].

2 Mathematical Properties

A random variable X is said to have transmuted powered moment exponential probability distribution denoted by TPME λ with parameters $\alpha, \beta, \lambda > 0$ and $-1 \le \lambda \le 1$, if its CDF and PDF are given by

$$G_1(x) = (1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right] - \lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^2 \tag{2.1}$$

$$g_1(x) = \alpha\beta^2 x^{2\alpha-1} e^{-\beta x^{\alpha}}\left[(1+\lambda) - 2\lambda\left\{1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right\}\right] \qquad \alpha, \beta, x > 0 \tag{2.2}$$

where α is the shape parameter, β is location parameter and λ the transmuting parameter, representing the different pattern of subject distribution

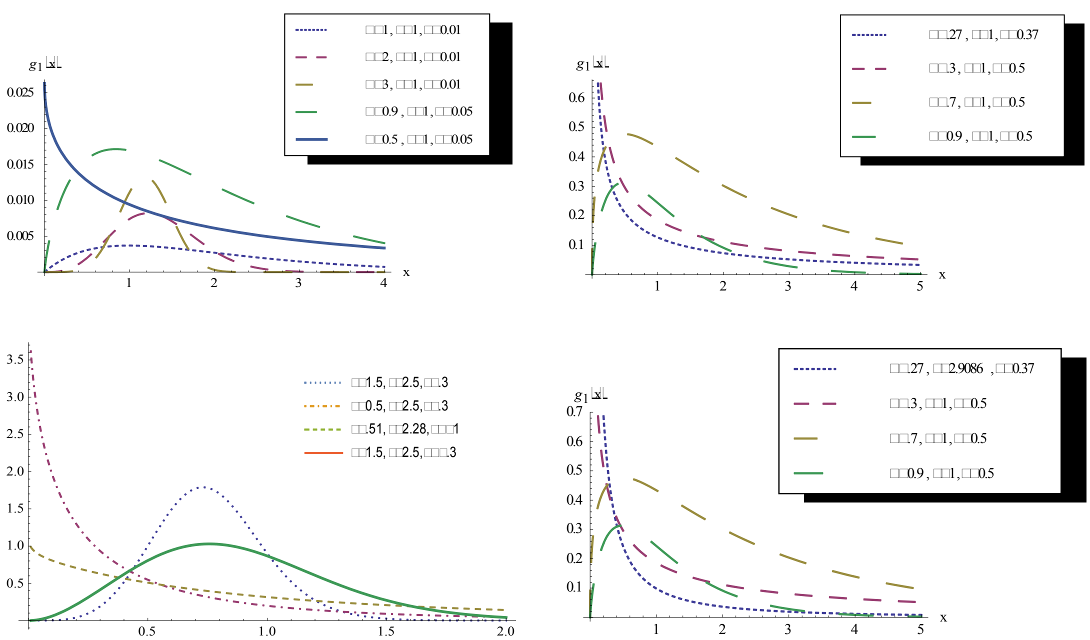


Fig. 1. Plots of PDF of transmuted powered moment exponential distribution for different values of parameter

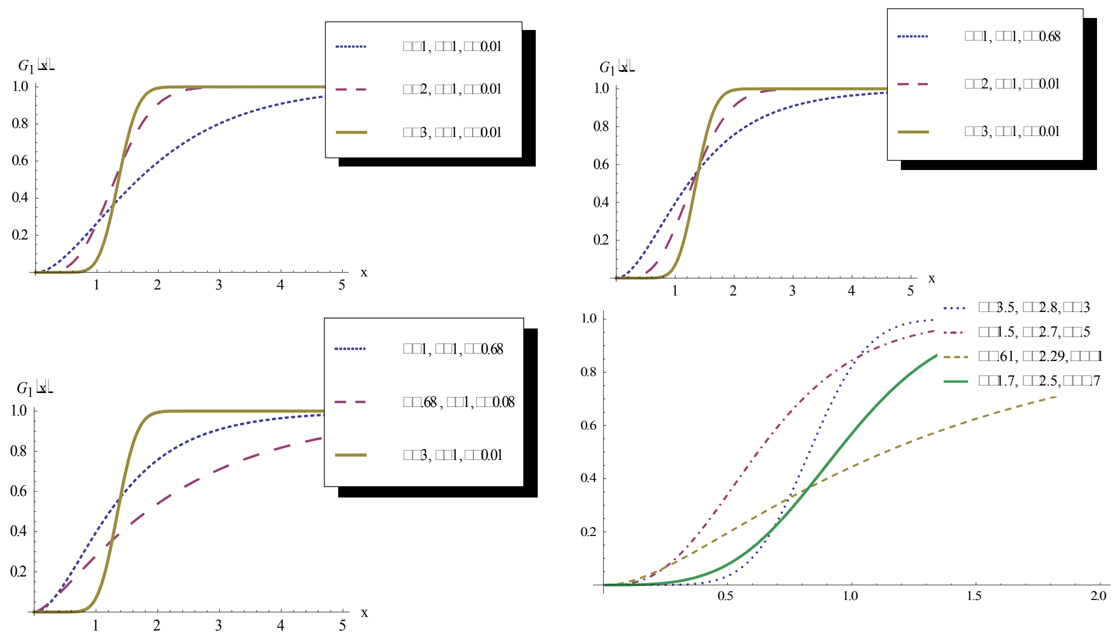


Fig. 2. Plots of CDF of transmuted powered moment exponential distribution for different values of parameter

2.1 Hazard rate function

If X has TPME$(x;\alpha,\beta,\lambda)$ distribution, then the hazard rate function is given by

$$h(x)=\frac{g_1(x)}{1-G_1(x)}=\frac{\alpha\beta^2x^{2\alpha-1}e^{-\beta x^\alpha}\left[(1+\lambda)-2\lambda\left\{1-\left(1+\beta x^\alpha\right)e^{-\beta x^\alpha}\right\}\right]}{1-(1+\lambda)[1-(1+\beta x^\alpha)e^{-\beta x^\alpha}]-\lambda[1-(1+\beta x^\alpha)e^{-\beta x^\alpha}]^2}$$

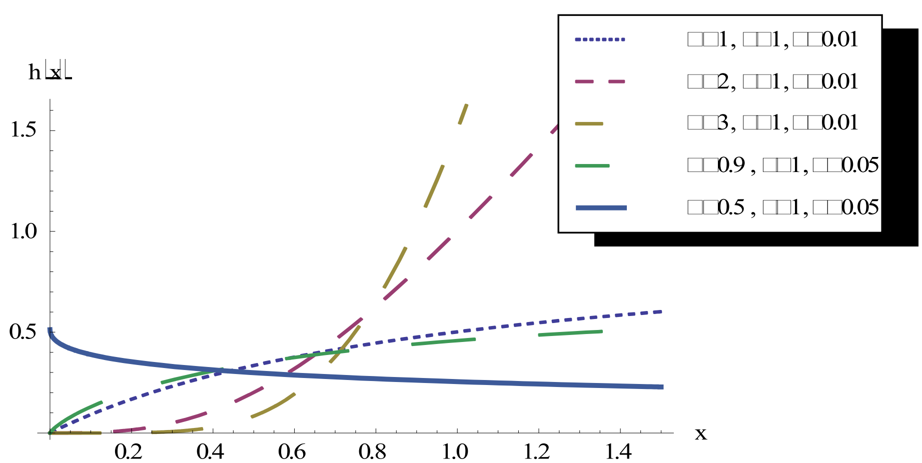


Fig. 3. Plots of *hrf* of transmuted powered moment exponential distribution for different values of parameter

2.2 Reverse hazard rate function

Reverse hazard rate function is defined as

$$R(x)=\frac{\alpha\beta^2 x^{2\alpha-1}e^{-\beta x^\alpha}[(1+\lambda)-2\lambda\{1-(1+\beta x^\alpha)e^{-\beta x^\alpha}}{1-((1+\lambda)[1-(1+\beta x^\alpha)e^{-\beta x^\alpha}]-\lambda[1-(1+\beta x^\alpha)e^{-\beta x^\alpha}]^2}$$

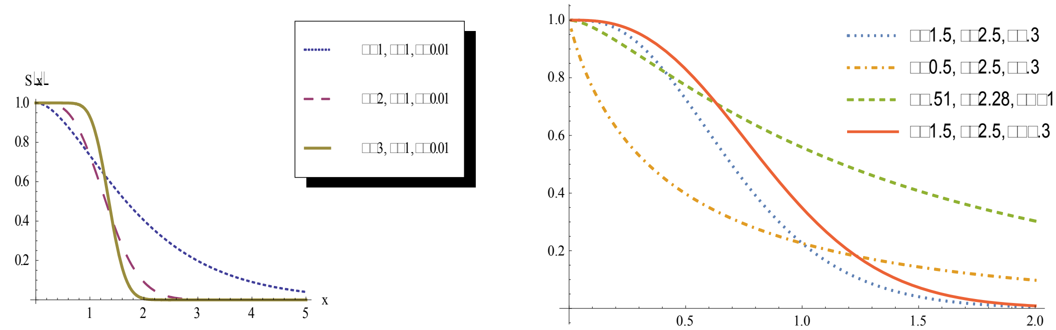


Fig. 4. Plots of reliability function of TPME distribution for different values of parameter

2.3 Moments and central Moments

If X has the TPME $(x;\alpha,\beta,\lambda)$ distribution with $|\lambda| \le 1$, then the r[th] moment of X is given as follows

$$E\left(X^r\right)=\frac{\Gamma\left(2+\frac{r}{\alpha}\right)}{\beta^{\frac{r}{\alpha}}}\left[(1+\lambda)-2\lambda+\frac{\lambda}{2^{\frac{r}{\alpha}+1}}+\frac{\lambda}{2^{\frac{r}{\alpha}+2}}\Gamma\left(2+\frac{r}{\alpha}\right)\right]$$

First four moments about origin can be respectively obtained by substituting $r=1,2,3,4$ in equation and are given as below

i. If $r=1$ then $E\left(X^1\right)=\frac{\Gamma\left(2+\frac{1}{\alpha}\right)}{\beta^{\frac{1}{\alpha}}}\left[(1+\lambda)-2\lambda+\frac{\lambda}{2^{\frac{1}{\alpha}+1}}+\frac{\lambda}{2^{\frac{1}{\alpha}+2}}\Gamma\left(2+\frac{1}{\alpha}\right)\right]$

ii. If $r=2$ then $E\left(X^2\right)=\frac{\Gamma\left(2+\frac{2}{\alpha}\right)}{\beta^{\frac{2}{\alpha}}}\left[(1+\lambda)-2\lambda+\frac{\lambda}{2^{2+\frac{2}{\alpha}}}\left\{1+\frac{1}{2}\Gamma\left(2+\frac{2}{\alpha}\right)\right\}\right]$

iii. If $r=3$ then $E\left(X^3\right)=\frac{\Gamma\left(2+\frac{3}{\alpha}\right)}{\beta^{\frac{3}{\alpha}}}\left[(1+\lambda)-2\lambda+\frac{\lambda}{2^{2+\frac{3}{\alpha}}}\left\{1+\frac{1}{2}\Gamma\left(2+\frac{3}{\alpha}\right)\right\}\right]$

iv. If $r=4$ then $E\left(X^4\right)=\dfrac{\Gamma\left(2+\frac{4}{\alpha}\right)}{\beta^{\frac{4}{\alpha}}}\left[\left(1+\lambda\right)-2\lambda+\dfrac{\lambda}{2^{2+\frac{4}{\alpha}}}\left\{1+\frac{1}{2}\Gamma\left(2+\frac{4}{\alpha}\right)\right\}\right]$

2.4 The variance, skewness and kurtosis

Measures can now be calculated using the following relationships

$$CV\left(X\right)=\sqrt{Var\left(X\right)}/E\left(X\right)$$

$$Skewness\left(X\right)=\mu_3/\mu_2^{\frac{3}{2}},$$

And $Kurtosis\left(X\right)=\dfrac{\mu_4}{\mu_2^2}$

Table 1. Mean, Variance, coefficient of variation, Skewness and kurtosis

	α = 0.68, β = 2 ,λ = - 0.5	α = 2, β = 2, λ = -0.5
m_1'	1.455	1.0361
m_2'	3.96	1.1875
m_3'	16.7499	1.4768
m_4'	99.7855	1.9687
$m_2=\sigma^2$	1.85	0.1139
m_3	1.55	-1.14
m_4	39.19	0.039
b_1	0.6172	-29.63
b_2	11.44	3.03

2.5 Moment generating function

If a random variable X follows TPME with the following PDF has the mgf as.

$$M_x\left(t\right)=\sum_{i=0}^{\infty}\frac{\left(t\right)^k}{k!}\times\frac{\Gamma\left(2+\frac{r}{\alpha}\right)}{\beta^{\frac{r}{\alpha}}}\left[\left(1+\lambda\right)-2\lambda+\frac{\lambda}{2^{1+\frac{r}{\alpha}}}+\frac{\lambda}{2^{2+\frac{r}{\alpha}}}\Gamma\left(2+\frac{r}{\alpha}\right)\right]$$

2.6 Quantile function

The quantile function can be found with the help of Lambert W function:

2.6.1 The Lambert W function:

The quantile of the TPME distribution is found by using Lambert function (The lambert function has attracted a great deal of attention beginning with Lambert (1758) and Euler (1799)), name Lambert W function.

Lambert W function has become a standard after its implementation in the computer algebra system Maple in the 1980s and subsequent publication by Corless et al. (1996) of a comprehensive survey of the history, theory and applications of this function. The Lambert W function is a multivalve complex function defined as the solution of the equation

$$W(z)\exp(W(z))=z$$

Where z is a complex number, if z is a real number such that $z \geq -\frac{1}{e}$ then $W(z)$ becomes a real function and there are two possible real branches. The real branch taking on values in $(-\infty,-1]$ is called the negative branch and denoted by W_{-1}. The real branch taking on values in $[-1,\infty)$ is called the principal branch and denoted by W_0.

$$(1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]-\lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^{2}=p$$

$\lambda u^{2}-(1+\lambda)u+p=0$,where $1-(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}=u$, the explicit function for quantile function can be found by using Lambert W function.

2.7 Maximum likelihood estimator of TPME

Let $x_1, x_2, ..., x_n$ be the random samples of size n from the TPME distribution. Then the log-likelihood function of is given by

$$\ln(l,\alpha,\beta,\lambda,x) = \ln\alpha + (2\alpha-1)\ln x - \beta x^{\alpha} - \lambda\beta x^{\alpha} - 2\ln\lambda + 2\ln\beta$$
$$+ (2\alpha-1)\ln x - \beta x^{\alpha} + 2\ln\lambda + 2\ln\beta + (2\alpha-1)\ln x$$

$$\frac{\partial \ln g_1(x)}{\partial\alpha} = \frac{1}{\alpha} + 2\ln x - \beta x^{\alpha}\ln x - \frac{2\lambda\beta x^{\alpha}\ln x[e^{-\beta x^{\alpha}} + \beta x^{\alpha}e^{-\beta x^{\alpha}} + 1]}{(1+\lambda) - 2\lambda[1-(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}}$$

Taking partial derivatives w.r.t to $\boldsymbol{\alpha}$, λ

$$\frac{\partial \ln g_1(x)}{\partial\beta} = \frac{2}{\beta} - x^{\alpha} + \frac{e^{-\beta x^{\alpha}}2\lambda\beta x^{\alpha}}{(1+\lambda) - 2\lambda[1-(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}}$$

$$\frac{\partial \ln g_1(x)}{\partial\lambda} = 1 - 2[1-(1+\beta x^{\alpha})e^{-\beta x^{\alpha}}]$$

$$\frac{\partial \ln g_1(x)}{\partial \alpha} = \frac{\partial \ln g_1(x)}{\partial \beta} = \frac{\partial \ln g_1(x)}{\partial \lambda} = 0$$

The value of estimates of $\boldsymbol{\alpha}$, and λ can never be estimates analytically. The MLE (Maximum likelihood Estimate) is obtained by solving the non-linear system. The solution of this non-linear system of equations does not have a closed form, but can be found numerically by using software such as MATHEMATICA, MAPLE and R.

2.8 Vitality function

The vitality function of the *pdf* (2.2) is defined as:

$$\mathrm{V}(x) = \mathrm{E}\left[\mathrm{X} \mid \mathrm{X} > x\right]$$

$$V(x) = \frac{1}{\beta^{\frac{1}{\alpha}}}\left[(1-\lambda)\left(\Gamma(\frac{1}{\alpha}+2), \beta x^{\alpha}\right) - \frac{1}{2^{\frac{1}{\alpha}+2}}\left(\Gamma(\frac{1}{\alpha}+2), 2\beta x^{\alpha}\right) - \frac{1}{2^{\frac{1}{\alpha}+3}}\left(\Gamma(\frac{1}{\alpha}+3), \beta x^{\alpha}\right)\right]$$

2.9 Mean residual function

The mean residual function of the *pdf* (2.2) is defined as:

$$m(x) = \frac{1}{\overline{G_1}(x)} \int_X^{\infty} \overline{G_1}(t)\, dt$$

$$= \frac{\int_x^{\infty} \alpha \beta^2 t^{2\alpha-1} e^{-\beta x^{\alpha}} \left[(1+\lambda) - 2\lambda\left\{1 - \left(1+\beta t^{\alpha}\right) e^{-\beta t^{\alpha}}\right\}\right] dt}{1-(1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right] - \lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^2}$$

$$= \frac{\left[\left(\Gamma \frac{1}{\alpha}, \beta x^{\alpha}\right) + \left(\Gamma(\frac{1}{\alpha}+1), \beta x^{\alpha}\right)\right]}{\alpha \beta^{\frac{1}{\alpha}} \left(1+\beta x^{\alpha}\right) e^{-\beta x^{\alpha}}}$$

2.10 Scaled total Time function

The scaled total time of the *pdf* (2.1) is defined as:

$$S_F(x) = \frac{1}{E(x)} \int_0^x \overline{G_1}(t)\, dt$$

$$= \frac{1 - \int_x^{\infty} \alpha \beta^2 t^{2\alpha-1} e^{-\beta x^{\alpha}} \left[(1+\lambda) - 2\lambda\left\{1-\left(1+\beta t^{\alpha}\right)e^{-\beta t^{\alpha}}\right\}\right] dt}{E(x)}$$

$$=\frac{\frac{1}{\alpha\beta^{\alpha}}\left[\left(\Gamma\frac{1}{\alpha},\beta x^{\alpha}\right)-\left(\Gamma(1+\frac{1}{\alpha}),\beta x^{\alpha}\right)\right]}{\frac{\Gamma(2+\frac{1}{\alpha})}{\beta^{\frac{1}{\alpha}}}\left[(1+\lambda)-2\lambda+\frac{\lambda}{2^{\frac{1}{\alpha}+1}}\left\{1+\frac{1}{2}\Gamma\left(2+\frac{1}{\alpha}\right)\right\}\right]}$$

2.11 Order Statistics

The order statistics mostly appear in the problems of the estimation and testing. The application of extreme values is very common in reliability, meteorology, econometrics and various areas of research.

The PDF $g_{\text{TPME};X_{(j)}}(x)$ of jth order statistic $X_{(j)}$ is

$$g_{\text{TPME};X_{(j)}}(x)=\frac{1}{B(\text{j},n-j+1)}g_{\text{TPME}}(x)\left[G_{\text{TPME}}(x)\right]^{j-1}\left[1-G_{\text{TPME}}(x)\right]^{n-j}.,$$

The PDF $g_{\text{TPME};X_{(j)}}(x)$ of jth order statistic $X_{(j)}$ TPME$(x;\alpha,\beta,\lambda)$ distribution is given by ‘’

$$g_{TPME;X_{(j)}}(x)=\frac{1}{B(\text{j},n-j+1)}\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{n-j}{\ell}\left\{(1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]-\lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^{2}\right\}^{\ell+j-1}\alpha\beta^{2}x^{2\alpha-1}e^{-\beta x^{\alpha}}\left[(1+\lambda)-2\lambda\left\{1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right\}\right]$$

The PDF $g_{\text{TPME};X_{(n)}}(x)$ of nth order statistic $X_{(n)}$ is $g_{1\text{TPME};X_{(n)}}(x)=n\left[G_{1\text{TPME}}(x)\right]^{n-1}G_{1\text{TPME}}(x).$

The PDF $g_{\text{TPME};X_{(n)}}(x)$ of nth order statistic $X_{(n)}$ for TPME$(x;\alpha,\beta,\lambda)$ distribution is given by

$$g_{X_{n:n}}(x)=n\left[(1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]-\lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^{2}\right]^{n-1}\alpha\beta^{2}x^{2\alpha-1}e^{-\beta x^{\alpha}}\left[(1+\lambda)-2\lambda\left\{1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right\}\right].$$

The PDF $g_{\text{TPME};X_{(1)}}(x)$ of $1st$ order statistic $X_{(1)}$ is

$$g_{\text{TPME};X_{(1)}}(x)=n\left[1-G_{\text{TPME}}(x)\right]^{n-1}G_{\text{TPME}}(x).$$

The PDF $g_{\text{TPME};X_{(1)}}(x)$ of $1st$ order statistic $X_{(1)}$ for TPME$(x;\alpha,\beta,\lambda)$ distribution is given by

$$g_{X_{1:n}}(x)=n\left[1-(1+\lambda)\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]+\lambda\left[1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right]^{2}\right]^{n-1}$$
$$\times\alpha\beta^{2}x^{2\alpha-1}e^{-\beta x^{\alpha}}\left[(1+\lambda)-2\lambda\left\{1-\left(1+\beta x^{\alpha}\right)e^{-\beta x^{\alpha}}\right\}\right].$$

3 Simulation Study

We adopt the Monte Carlo simulation study to access the performance of the MLE's of $\Theta = (\alpha, \beta, \lambda)$ through Mathematica 10.2 version. We generate different n sample observation from the quantile function above of the model TPME distribution. The parameters are estimated by maximum likelihood method. We considered different sample size = 20, 30, 50, 100, 300 and 500 and the number of repetition is 10000. The true parameters value as α, β, λ with three different sets of values, in Tables 2 and 3, of below shows the bias and mean squared error (MSE) of the estimate parameters at different parameter values. We observed that, when we increase sample sizes "n" the bias and Mean square error for the TPME model given below as: (α, β, λ) decreases with respect to the best estimation.

Table 2. The Bias and MSE on Monte Carlo simulation for parameters values of TPMED

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=20$	2.2531	0.2531	1.1341
		$n=30$	2.2401	0.2401	1.0914
		$n=50$	2.2032	0.2032	0.9912
		$n=100$	2.1352	0.1352	0.9355
		$n=300$	2.0917	0.0917	0.6225
		$n=500$	2.0039	0.0039	0.4015
β	3	$n=20$	3.2641	0.2641	0.9845
		$n=30$	3.2324	0.2324	0.8434
		$n=50$	3.2131	0.2131	0.7694
		$n=100$	3.2015	0.2015	0.7215
		$n=300$	3.0636	0.0636	0.6319
		$n=500$	3.0419	0.0419	0.2726
λ	3	$n=20$	3.3215	0.3215	0.8624
		$n=30$	3.2525	0.2525	0.8117
		$n=50$	3.1849	0.1849	0.7019
		$n=100$	3.1219	0.1219	0.6442
		$n=300$	3.1514	0.1514	0.4610
		$n=500$	3.0323	0.0323	0.1112

Table 3. The Bias and MSE on monte carlo simulation for parameters values for TPMED

Parameter	True value	Sample size n	Mean	Bias	MSE
α	2	$n=20$	2.2885	0.2885	0.9212
		$n=30$	2.2532	0.2532	0.8734
		$n=50$	2.2475	0. 2475	0.8578
		$n=100$	2.1238	0.1238	0.7296
		$n=300$	2.0832	0.0832	0.3657
		$n=500$	2.0105	0.0105	0.1747
β	3	$n=20$	3.3184	0.3184	1.0413
		$n=30$	3.2701	0.2701	0.9131
		$n=50$	3.2268	0.2268	0.8264
		$n=100$	3.1993	0.1993	0.7462
		$n=300$	3.1234	0.1234	0.4319
		$n=500$	2.9826	-0.0174	0.1135
λ	0.5	$n=20$	0.6821	0.1821	0.3764
		$n=30$	0.6674	0.1674	0.3426
		$n=50$	0.6521	0.1521	0.3215
		$n=100$	0.5523	0.0523	0.1269
		$n=300$	0.5176	0.0176	0.1145
		$n=500$	0.5069	0.0069	0.0285

Given first three sample moments, the corresponding $\Theta = (\alpha, \beta, \lambda)$ values are estimated from the actual theoretical first three population moments derived from (The sampling distributions of estimated $\Theta = (\alpha, \beta, \lambda)$ are given in Table 4 based on various sample sizes. For small samples, the percentage of estimates falling in the indicated interval increases with larger sample size. Using this range, we estimate Θ by the method of moments. If we include omitted data, we expect larger Mean Square Error (MSE). This MSE, however, decreases with increasing sample size

Table 4. Percentage of sample estimates of $\Theta = (\alpha, \beta, \lambda)$ through method of moments (MM) for the TPME model

N	% estimated values of parameter in indicated interval with $\alpha = 2$	% estimated values of parameter in indicated interval with $\beta = 3$	% estimated values of parameter in indicated interval with $\lambda = 0.5$
	$1.4 < \hat{\alpha} < 2.6$	$2.5 < \hat{\beta} < 3.5$	$0.3 < \hat{\lambda} < 0.7$
30	88.68%	85.28%	80.52%
50	92.64%	91.26%	86.52%
100	97.45%	94.94%	89.71%
250	98.02%	97.62%	94.76%
500	99.64%	99.23%	96.89%

Table 5. Percentage of sample estimates of $\Theta = (\alpha, \beta, \lambda)$ through method of moments (MM) for theTPME model

N	% estimated values of parameter in indicated interval with $\alpha = 2$	% estimated values of parameter in indicated interval with $\beta = 3$	% estimated values of parameter in indicated interval with $\lambda = 0.5$
	$1.4 < \hat{\alpha} < 2.6$	$2.5 < \hat{\beta} < 3.5$	$0.3 < \hat{\lambda} < 0.7$
30	89.67%	88.38%	83.12%
50	95.32%	93.45%	86.34%
100	98.67%	95.14%	89.67%
250	99.12%	98.62%	97.25%
500	99.89%	99.61%	98.37%

4 Application

In this section, the flexibility of some special models of *TPME* is examined using three real data sets. We illustrate the superiority of new selected distribution as compared with some sub-models.

Based on the maximum-likelihood method, the unknown parameters of each distribution are estimated. Some selected measures as; Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), the correct Akaike information criterion (*CAIC*), and the Kolmogorov-Smirnov (*k-s*) are obtained to compare the fitted models (as seen in Table 1). The mathematical form of these measures is as follows:

$$AIC = 2k - 2\ln L, \qquad CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k\ln(n) - 2\ln L,$$

Where k is the number of models parameter, n is the sample size and $\ln L$ is the maximized value of the log-likelihood function under the fitted models.

Also, $k-s=\sup_y\left[F_n(y)-F(y)\right]$, where $F_n(y)=\frac{1}{n}$ (number of observation $\leq y$), and $F(y)$ denotes the cdf. The best distribution is the distribution corresponding to the lower values of, *AIC, AICC,-2ln l*, and *k-s* statistics. The results for mentioned measures for all models are reported in Tables.

4.1 First data set

The first data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:

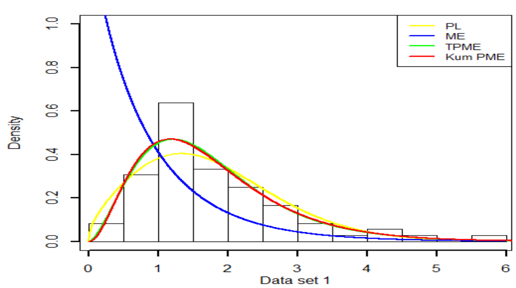
0.1, 0.33, 0.44, 0.56, 0.59, 0.59, 0.72, 0.74, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55, 2.54, 0.77.

Table 6. Criteria for comparison for first data set

Model	$k-s$	*AIC*	*CAIC*	**-2lnL**
TPME	0.075	194.35	194.70	188.354
PL	0.084	196.796	196.85	192.66
PME	0.352	254.11	254.18	250.0116
KSPME	0.0843	196.89	197.4	188.796

For the first data set, the values of *k-s,AIC*, *BIC* and *CAIC* are record in Table 6.

The plots of the estimated cumulative and estimated densities of the fitted models are achieved in Figs.5 and 6 respectively.



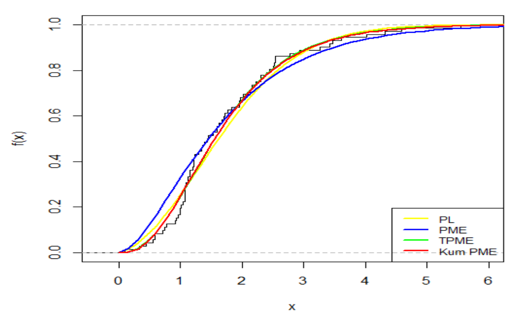


Fig. 5. Plots of PDF and CDF of transmuted powered moment exponential distribution for first data set

4.2 2nd Data set

The data was extracted from (Abdul-Moniem and Seham 2015) and it has previously been used by Barlow et al., (1984), the data is as follows; Owoloko et al. Springer Plus (2015) 4:818 DOI 10.1186/s40064-015-1590-6

0.0251,	0.0886,	0.0891,	0.2501,	0.3113,	0.3451,	0.4763,	0.5650,
0.5671,	0.6566,	0.6748,	0.6751,	0.6753,	0.7696,	0.8375,	0.8391,
0.8425,	0.8645,	0.8851,	0.9113,	0.9120,	0.9836,	1.0483,	1.0596,
1.0773,	1.1733,	1.2570,	1.2766,	1.2985,	1.3211,	1.3503,	1.3551,
1.4595,	1.4880,	1.5728,	1.5733,	1.7083,	1.7263,	1.7460,	1.7630,
1.7746,	1.8275,	1.8375,	1.8503,	1.8808,	1.8878,	1.8881,	1.9316,
1.9558,	2.0048,	2.0408,	2.0903,	2.1093,	2.1330,	2.2100,	2.2460,
2.2878,	2.3203,	2.3470,	2.3513,	2.4951,	2.5260,	2.9911,	3.0256,
3.2678,	3.4045,	3.4846,	3.7433,	3.7455,	3.9143,	4.8073,	5.4005,
5.4435,	5.5295,	6.5541,	9.0960.				

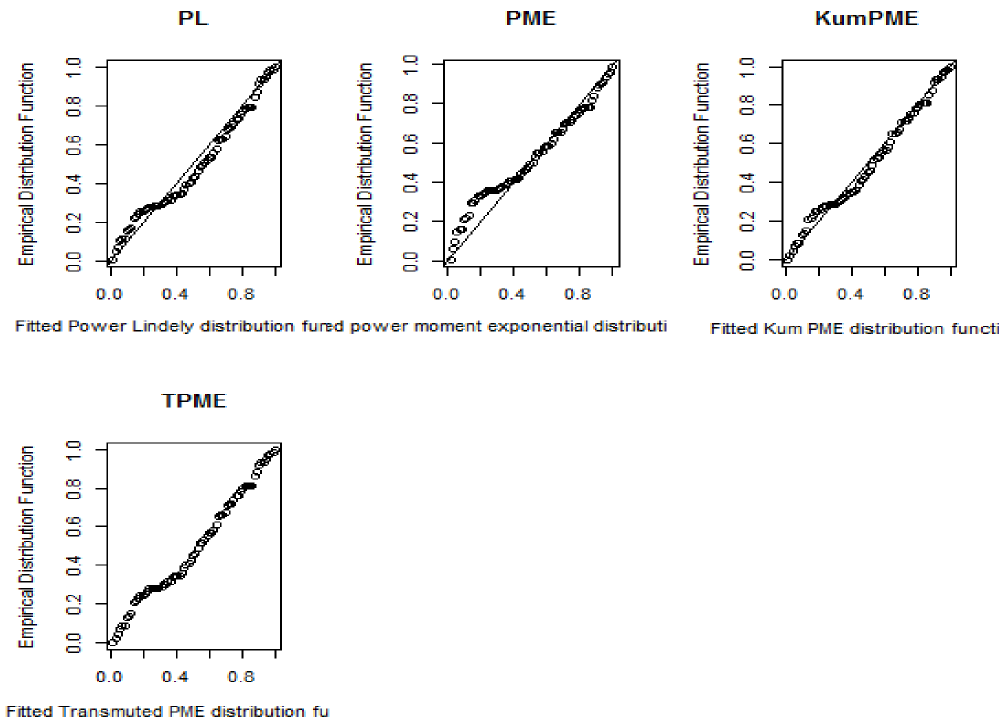


Fig. 6. The Q-Q plots for first data set

Table 7. Criteria for comparison for 2nd data set

Model	$k-s$	*AIC*	*CAIC*	**-2lnL**
TPME	0.065	193.35	193.70	178.254
PL	0.094	195.796	197.85	194.66
PME	0.452	253.11	255.18	249.0116
KSPME	0.0943	197.89	198.4	198.796

For the 2nd data set, the values of *k-s*,*AIC*, *BIC* and *CAIC* are record in Table 7.

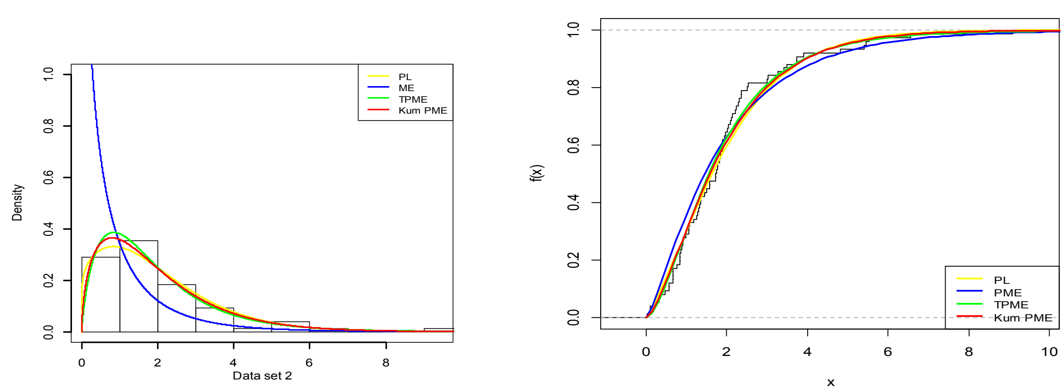


Fig. 7. Plots of PDF and CDF of transmuted powered moment exponential distribution for 2nd data set

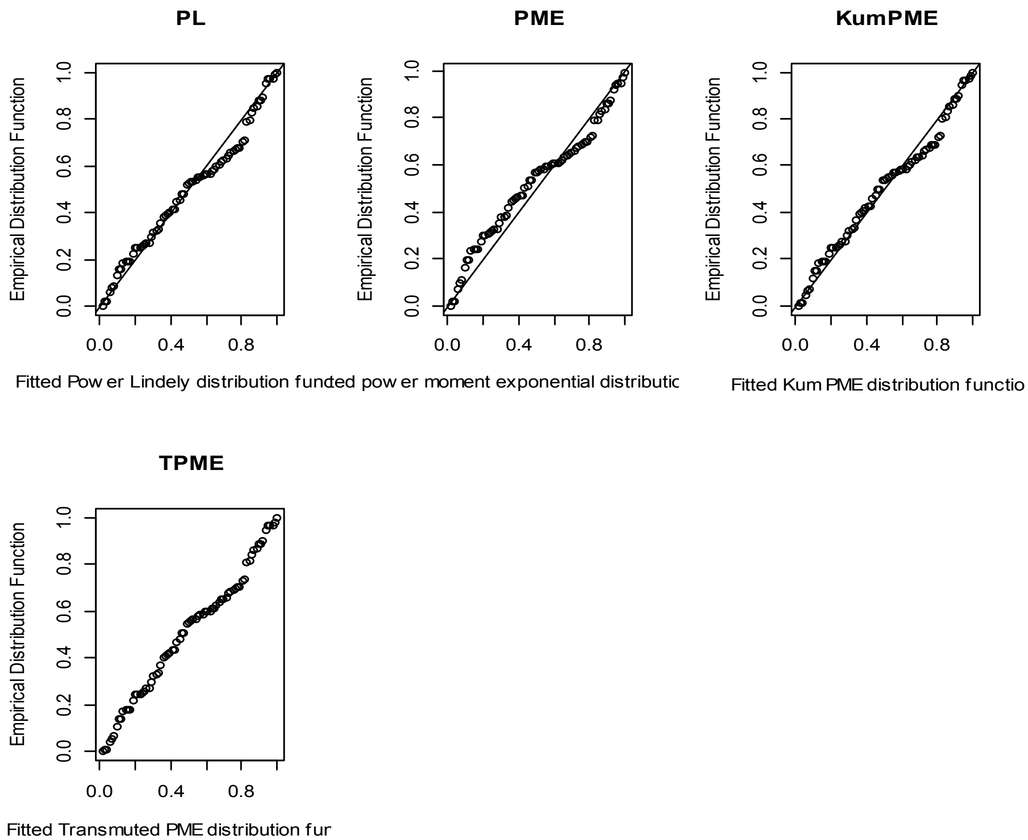


Fig. 8. The Q-Q plots for 2nd data set

Table 8. The eztimeted values of parameters with respective S.E

	Estimated values	**S.E**
$\hat{\alpha}$	1.2840	0.1161
$\hat{\beta}$	0.7269	0.1472
$\hat{\lambda}$	0.5706	0.4098

5 Conclusion

We introduce a new class of lifetime models called the transmuted powered moment exponential distribution**.** More specifically, the transmuted powered moment exponential distribution covers several new distributions. Various basic properties of TPME distribution are derieved.Survival analysis including survival function, hazard rate function and other related measures are computed. Parameters are estimated using the technique of Maximum Likelihood Estimation (MLE) [24-31].The new distribution was applied to a real data set. This model provides a better fit than several other related models. Also, mathematical properties of the new family, including expressions for density function, moments, moment generating function, quantile function, are provided. The hazard rate function has various shapes such as constant, increasing, decreasing, and bathtub. By simulation procedures it is discovered that the ML estimators are consistent since the bias and MSE approach to zero when the sample size increases. The usefulness of the model associated with this distribution is illustrated by two real data sets and the new model provides a better fit than the models provided in literature.

Competing Interests

Authors have declared that no competing interests exist.

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